# CBMS Lecture 6

Alan E. Gelfand
Duke University

# Multivariate spatial modeling

- Point-referenced spatial data often come as multivariate measurements at each location
- Examples:
  - Environmental monitoring stations yield measurements on ozone, NO, CO, $PM_{2.5}$, etc.
  - In atmospheric modeling at a given site we observe surface temperature, precipitation and wind speed
  - At a monitoring site we observe precipitation, wet sulfate deposition, wet nitrate deposition
  - At locations in a forest, we observe tree growth, soil moisture, light availability, climate variables
  - In real estate modeling for a commercial property we observe selling price and total rental income
- We anticipate dependence between measurements
  - at a particular location
  - across locations

# Basic issues

- $\mathbf{Y}(\mathbf{s})$ denotes a $p \times 1$ vector of random variables at $\mathbf{s}$
- We seek to model $\mathbf{Y}(\mathbf{s}) : \mathbf{s} \in D$, again specifying finite dimensional distributions, e.g., for $\mathbf{Y} = (\mathbf{Y}(\mathbf{s}_1), \ldots, \mathbf{Y}(\mathbf{s}_n))$
- *Crucial object:* the cross-covariance

$$C(\mathbf{s}, \mathbf{s}') = Cov(\mathbf{Y}(\mathbf{s}), \mathbf{Y}(\mathbf{s}'))$$

  a $p \times p$ matrix that need not be symmetric, i.e., $cov(Y_j(\mathbf{s}), Y_{j'}(\mathbf{s}'))$ need not equal $cov(Y_{j'}(\mathbf{s}), Y_j(\mathbf{s}'))$
- $C(\mathbf{s}, \mathbf{s}')$ is not positive definite except in a limiting sense: $C(\mathbf{s}, \mathbf{s})$ is the covariance matrix associated with $\mathbf{Y}(\mathbf{s})$.
- Our primary focus: Gaussian processes and valid specification for $C(\mathbf{s}, \mathbf{s}')$ to overlay

# Separable models

- A common specification is the separable model

$$C(\mathbf{s}, \mathbf{s}') = \rho(\mathbf{s}, \mathbf{s}') \cdot T$$

  where $\rho$ is a valid (univariate) correlation function and $T$ is a $p \times p$ positive definite matrix
- $T$ is the non-spatial or "local" covariance matrix
- $\rho$ controls spatial association based upon proximity
- Easy to verify that $\Sigma_{\mathbf{Y}} = H \otimes T$, where $H_{ij} = \rho(\mathbf{s}_i, \mathbf{s}_j)$ and $\otimes$ is the Kronecker product.
  - $\Sigma_{\mathbf{Y}}$ is positive definite since $H$ and $T$ are
  - $\Sigma_{\mathbf{Y}}$ is convenient since $|\Sigma_{\mathbf{Y}}| = |H|^p \, |T|^n$ and $\Sigma_{\mathbf{Y}}^{-1} = H^{-1} \otimes T^{-1}$.

# Application: Bivariate spatial regression

- A single covariate $X(\mathbf{s})$ and a univariate response $Y(\mathbf{s})$
- Treat this as a bivariate process. (WHY?)

$$\mathbf{Z}(\mathbf{s}) = \begin{pmatrix} X(\mathbf{s}) \\ Y(\mathbf{s}) \end{pmatrix} \sim N(\boldsymbol{\mu}(\mathbf{s}), T)$$

- Simplifying assumptions:
  - Separable cross-covariance for $\mathbf{Z}(\mathbf{s})$
  - $\boldsymbol{\mu}(\mathbf{s}) = (\mu_1, \mu_2)$, i.e., constant means.
- Then, $p(Y(\mathbf{s})|X(\mathbf{s})) = N(\beta_0 + \beta_1 X(\mathbf{s}), \sigma^2)$ where:

$$\beta_0 = \mu_2 - \frac{T_{12}}{T_{11}}\mu_1,\ \beta_1 = \frac{T_{12}}{T_{11}},\ \text{and}\ \sigma^2 = T_{22} - \frac{T_{12}^2}{T_{11}}$$

- Regression model parameters are functions of process model parameters

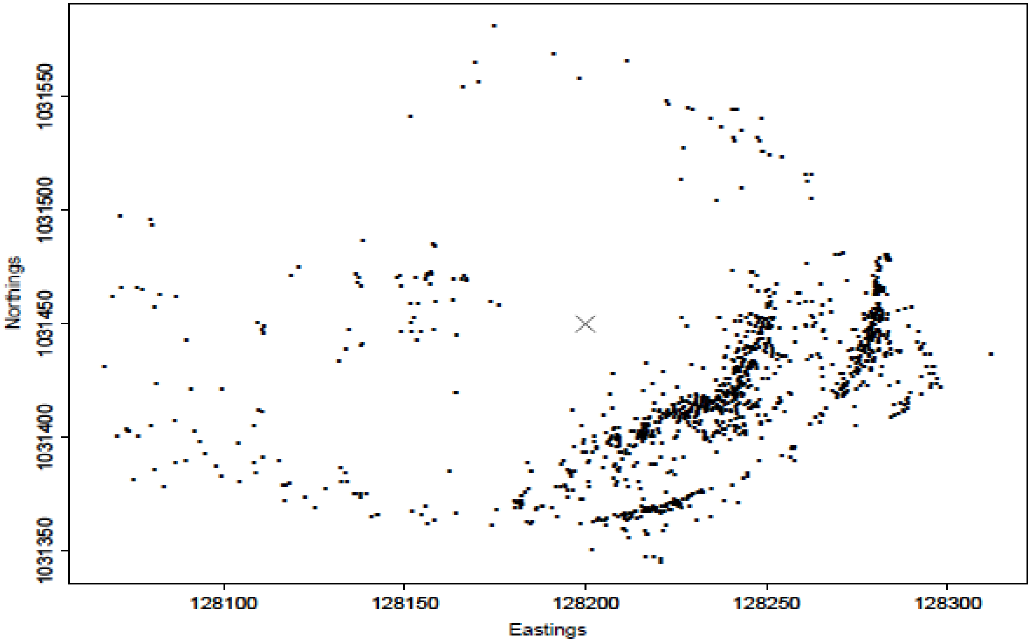
# Bivariate spatial regression (cont'd)

- Rearrangement of the components of $\mathbf{Z}$ to $\widetilde{\mathbf{Z}} = (X(\mathbf{s}_1), X(\mathbf{s}_2), \ldots, X(\mathbf{s}_n), Y(\mathbf{s}_1), Y(\mathbf{s}_2), \ldots, Y(\mathbf{s}_n))'$ yields
$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \sim N\left( \begin{pmatrix} \mu_1 \mathbf{1} \\ \mu_2 \mathbf{1} \end{pmatrix}, \; T \otimes H(\phi) \right),$$
- Priors: Wishart for $T^{-1}$, vague but proper normal for $(\mu_1, \mu_2)$, discrete prior for $\phi$
- Full conditionals for Gibbs sampler: again Wishart for $T^{-1}$, bivariate normal for $(\mu_1, \mu_2)$; sampling from a discrete distribution for $\phi$ or perhaps a uniform on (0, .5max dist)

# Dew-shrub data example

- 1129 locations with UTM coordinates
- $Y(\mathbf{s})$ : shrub density at location $\mathbf{s}$
- $X(\mathbf{s})$ : Dew duration at location $\mathbf{s}$
- Illustrative analysis assuming separability and an exponential correlation function, $\rho(h;\phi) = e^{-\phi h}$
- Conjugate priors for $\boldsymbol{\mu}$, $T$ as above; prior for $\phi$ has infinite variance and suggests a range $(3/\phi)$ of 125 km, roughly half the maximum pairwise distance in the region
- $(\mu_1, \mu_2, T_{11}, T_{12}, T_{22})$ updated directly; $\phi$ updated via Metropolis
- Posterior samples of $(\beta_0, \beta_1, \sigma^2)$ from posterior samples of process parameters

# Sites yielding dew and shrub data



# Parameter estimation, dew-shrub data

| Parameter | 2.5% | 50% | 97.5% |
|---|---|---|---|
| $\mu_1$ | 73.12 | 73.89 | 74.67 |
| $\mu_2$ | 5.20 | 5.38 | 5.572 |
| $T_{11}$ | 95.10 | 105.22 | 117.69 |
| $T_{12}$ | –4.46 | –2.42 | –0.53 |
| $T_{22}$ | 5.56 | 6.19 | 6.91 |
| $\phi$ | 0.01 | 0.03 | 0.21 |
| $\beta_0$ | 5.72 | 7.08 | 8.46 |
| $\beta_1$ | –0.04 | –0.02 | –0.01 |
| $\sigma^2$ | 5.58 | 6.22 | 6.93 |
| $T_{12}/\sqrt{T_{11}T_{22}}$ | –0.17 | –0.10 | –0.02 |

$\Rightarrow$ Surprising - a significant negative association between dew duration and shrub density!

# Benefits and limitations of separability

- Benefits:
  - Easy interpretation (decomposition of variance structure)
  - Substantial computational benefits
- Limitations:
  - Symmetry in cross-covariance matrix (not so serious)
  - Imposes same spatial range for every component (more serious, only one correlation function)
- a proposed solution
  - Coregionalization models

# An simple nonseparable example

- The delay effect or pure offset model
- Define $\mathbf{Y}(\mathbf{s})$ to be two-dimensional such that $Y_2(\mathbf{s}) = Y_1(\mathbf{s} + \boldsymbol{\lambda})$. $\boldsymbol{\lambda}$ is a *delay* vector
- Cross covariance matrix is

$$\begin{pmatrix} \sigma^2\rho(h) & \sigma^2\rho(h+\lambda) \\ \sigma^2\rho(-h+\lambda) & \sigma^2\rho(h) \end{pmatrix}$$

- Here $\rho$ is valid
- Can add a nugget, i.e., define $Y_2(\mathbf{s}) = Y_1(\mathbf{s} + \boldsymbol{\lambda}) + \epsilon(\mathbf{s})$
- Potential application to exposures driven by wind direction

# Linear Model of Coregionalization

- For point referenced data, $\mathbf{Y}(\mathbf{s}) = A\mathbf{w}(\mathbf{s})$ where $\mathbf{w}(\mathbf{s}) = (w_1(\mathbf{s}), w_2(\mathbf{s}), \ldots, w_p(\mathbf{s}))$
- $p$ independent spatial processes with stationary correlation functions $\rho_j(\mathbf{s} - \mathbf{s}'), j = 1, 2, \ldots, p$
- If $\rho_j = \rho$ for all $j \Longleftrightarrow$ separable case with $AA' = T$
- In general, the cross covariance matrix is (with $\mathbf{a}_j$ being the columns of $A$)

$$C(\mathbf{s} - \mathbf{s}') = \sum_{j=1}^{p} \rho_j(\mathbf{s} - \mathbf{s}')\mathbf{a}_j\mathbf{a}_j'$$

- Approach is "constructive" so $C(\mathbf{s} - \mathbf{s}')$ immediately valid, still stationary, and provides a distinct covariance function for each component

# Linear Model of Coregionalization

- More general: $\mathbf{Y}(\mathbf{s}) = A(\mathbf{s})\mathbf{w}(\mathbf{s})$.
  A spatially varying LMC!
- model $A(\mathbf{s}) \Leftrightarrow$ model $T(\mathbf{s}) = A(\mathbf{s})A'(\mathbf{s})$
- Possibilities for $T(\mathbf{s})$:
  - $T(\mathbf{s}) = g(X(\mathbf{s})) \times T$
  - $T(\mathbf{s})$ is a spatial process (e.g., $T^{-1}(\mathbf{s})$ is a spatial Wishart process)
- Computationally demanding

# cont.

- Specification of A
- $p \times p$ entries in $A$ but, since $A \Leftrightarrow T$, only require $\frac{p(p+1)}{2}$ parameters. For convenience, we often take $A$ to be lower triangular.
- Given $\phi_1, \cdots, \phi_p$, the cross covariance matrix is symmetric, regardless of $A$.
- Number of parameters in the model $\frac{p(p+1)}{2} + pm$ where $m$ is the dimension of $\phi_j$, i.e., number of parameters in the individual correlation functions.
- With $p = 2$, we have 3 parameters in $A$ and, using an exponential covariance function, $m = 2$ decay parameters

## cont.

- The one-to-one relationship between $T$ and lower triangular $A$ is standard.
- When $p = 2$ we have

$$a_{11} = \sqrt{T_{11}},\ a_{21} = \frac{T_{12}}{\sqrt{T_{11}}},\ a_{22} = \sqrt{T_{22} - \frac{T_{12}^2}{T_{11}}}$$

- When p=3 we add

$$a_{31} = \frac{T_{13}}{\sqrt{T_{11}}},\quad a_{32} = \frac{T_{11}T_{23} - T_{12}T_{13}}{T_{11}(T_{11}T_{22} - T_{12}^2)}$$

$$\text{and } a_{33} = \sqrt{T_{33} - \frac{T_{13}^2}{T_{11}} - \frac{(T_{11}T_{23} - T_{12}T_{13})^2}{T_{11}(T_{11}T_{22} - T_{12}^2)}}$$

# cont.

- More explicitly

$$\begin{pmatrix} Y_1(\mathbf{s}) \\ Y_2(\mathbf{s}) \\ \vdots \\ Y_j(\mathbf{s}) \\ \vdots \\ Y_p(\mathbf{s}) \end{pmatrix} = \begin{pmatrix} a_{11} w_1(\mathbf{s}) \\ a_{21} w_1(\mathbf{s}) + a_{22} w_2(\mathbf{s}) \\ \vdots \\ \sum_{l=1}^{j} a_{jl} w_l(\mathbf{s}) \\ \vdots \\ \sum_{l=1}^{p} a_{jl} w_l(\mathbf{s}) \end{pmatrix} .$$

- $\mathbf{Y}(\mathbf{s})$ is stationary, has a symmetric cross-covariance matrix, with a *different* variance and, if the $\rho(.;\phi_j)$'s are isotropic, a *different* range for each component of $\mathbf{Y}(\mathbf{s})$.

# General Multivariate Spatial Model

- So, we arrive at the model

$$\mathbf{Y}(\mathbf{s}) = \boldsymbol{\mu}(\mathbf{s}) + \mathbf{v}(\mathbf{s}) + \boldsymbol{\epsilon}(\mathbf{s})$$

  with
- $\boldsymbol{\epsilon}(\mathbf{s}) \sim N(0, D_\epsilon)$, $(D_\epsilon)_{jj} = \tau_j^2$.
- $\mathbf{v}(\mathbf{s}) = A\mathbf{w}(\mathbf{s})$ following previous specification
- $w_j(\mathbf{s})$ are mean 0 Gaussian processes with individual correlation functions.
- $\boldsymbol{\mu}(\mathbf{s})$ arises from $\mu_j(\mathbf{s}) = \mathbf{X}_j^T(\mathbf{s})\boldsymbol{\beta}_j$.

# A useful example

- Spatially varying coefficient models (Gelfand et al., 2003)
- Model $Y(\mathbf{s}) = \mathbf{X}(\mathbf{s})^T \beta(\mathbf{s}) + \epsilon(\mathbf{s})$.
- Here $Y(\mathbf{s})$ is univariate. The multivariate process is for $\beta(\mathbf{s})$. Use coregionalization here.
- For $p = 2$, with $\mathbf{X}(\mathbf{s})$ having a column of "1"'s, we obtain $\beta_0(\mathbf{s}) + X(\mathbf{s})\beta_1(\mathbf{s})$
- Spatially varying intercept (like a spatial random effect) and a spatially varying slope.
- Analogous to longitudinal growth curve models
- A very rich class of *nonlinear* models
- Infer about the multivariate process is for $\beta(\mathbf{s})$ while only observing the univariate $Y(\mathbf{s})$ process

# Hierarchical Model

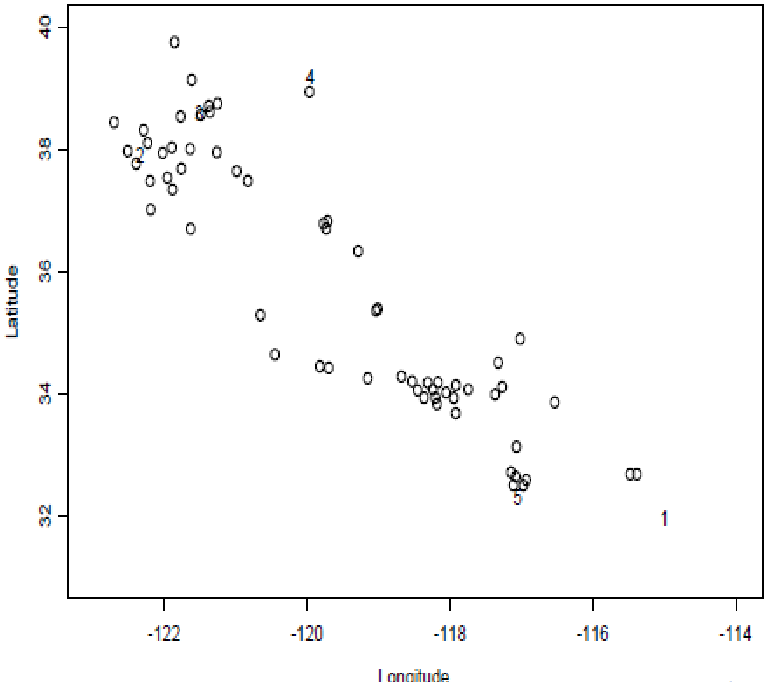
- 1st stage:
  $\mathbf{Y}(\mathbf{s}_i)|\{\boldsymbol{\beta}_j\}, \{\mathbf{v}(\mathbf{s}_i)\}, D_\epsilon \sim N(\boldsymbol{\mu}(\mathbf{s}_i) + \mathbf{v}(\mathbf{s}_i), D_\epsilon)$.
- 2nd stage:
  $$\mathbf{v} = \begin{pmatrix} \mathbf{v}(\mathbf{s}_1) \\ \vdots \\ \mathbf{v}(\mathbf{s}_n) \end{pmatrix} \sim N(\mathbf{0}, \textstyle\sum_{j=1}^{p} \mathbf{R}_j \otimes \mathbf{T}_j),$$
  $\mathbf{Y}(\mathbf{s}_i)$ into $\mathbf{Y}$, $\boldsymbol{\mu}(\mathbf{s}_i)$ into $\boldsymbol{\mu}$, marginalize over $\mathbf{v}$
  $$f(\mathbf{Y}|\{\boldsymbol{\beta}_j\}, D_\epsilon, \{\rho_j\}, T) =$$
  $$N\left(\boldsymbol{\mu}, \sum_{j=1}^{p}(H_j \otimes T_j) + I_{n\times n} \otimes D_\epsilon\right).$$
- 3rd stage: Priors on $\{\boldsymbol{\beta}_j\}$, $\{\tau_j^2\}$, $T$ and the parameters of the $\rho_j$.

# California Pollution Data Example

- From the California Air Resources Board. Available for download at `http://www.arb.ca.gov/aqd/aqdcd/aqdcddld.htm`.
- Daily average of Carbon Monoxide (CO), Nitric Oxide (NO) and Nitrogen dioxide ($NO_2$) based on hourly measurements on July, $6^{th}$, 1999 $\rightarrow$ **68 sites**.
- The observed correlations between these pollutants range from 0.46 (CO and NO) to 0.77 (NO and $NO_2$).
- Use the logarithm of the daily average of each of these variables.
- No information on covariates, such as temperature or wind directions, at these gauged sites.

# Monitoring stations

Location of the $63$ monitoring stations used to fit the mode
and the $5$ used for prediction of $NO_2$



# A model specification

- Can specify coregionalization model sequentially
- We anticipate smooth exposure surfaces so only spatial random effects, no nuggets
- The model:

$$CO(\mathbf{s}) = \mu_1 + \sigma_1 \tilde{w}_1(\mathbf{s})$$

$$NO(\mathbf{s})|CO(\mathbf{s}) = \mu_2 + \alpha CO(\mathbf{s}) + \sigma_2 \tilde{w}_2(\mathbf{s})$$

$$NO_2(\mathbf{s})|CO(\mathbf{s}), NO(\mathbf{s}) = \mu_3 + \gamma CO(\mathbf{s}) +$$

$$\beta NO(\mathbf{s}) + \sigma_3 \tilde{w}_3(\mathbf{s}),$$

with $\tilde{w}_j(\mathbf{s}) \sim GP(0, \rho_j)$ and
$\rho_j = \exp\{-\psi_j ||\mathbf{s} - \mathbf{s}'||\}$

# Prior specifications

$$\mu_1 \sim N(0,5),\ \mu_2 \sim N(0,5),\ \mu_3 \sim N(0,5)$$

$$\alpha \sim N(0,5), \gamma \sim N(0,0.2),\ \beta \sim N(0,0.2)$$

$$\sigma_1^2 \sim IG(5, 0.35 * 4),\ \sigma_2^2 \sim IG(5, 0.52 * 4),$$
$$\sigma_3^2 \sim IG(5, 0.13 * 4),$$

$$\psi_1 \sim Ga(0.6, 1),\ \psi_2 \sim Ga(0.6, 1),$$
$$\psi_3 \sim Ga(0.6, 1).$$

$p(\psi_j)$ based on $\psi = 3/\text{range}$ and range $=$ .5max dist

# Posterior Summaries

Posterior Summaries for CO (1), NO (2) and $NO_2$ (3).

| Parameter | Mean | 2.50% | Median | 97.50% |
|---|---|---|---|---|
| $\alpha$ | 0.296 | 0.045 | 0.292 | 0.553 |
| $\beta$ | 0.302 | 0.190 | 0.301 | 0.413 |
| $\gamma$ | 0.198 | 0.082 | 0.199 | 0.314 |
| $\mu_1$ | -0.922 | -1.135 | -0.921 | -0.710 |
| $\mu_2$ | -5.015 | -5.504 | -5.018 | -4.538 |
| $\mu_3$ | -2.602 | -3.281 | -2.602 | -1.943 |
| $\phi_1$ | 4.995 | 2.789 | 4.882 | 7.952 |
| $\phi_2$ | 2.186 | 1.064 | 2.081 | 3.854 |
| $\phi_3$ | 1.209 | 0.525 | 1.157 | 2.201 |
| $\sigma_1^2$ | 0.391 | 0.267 | 0.381 | 0.570 |
| $\sigma_2^2$ | 0.698 | 0.438 | 0.668 | 1.163 |
| $\sigma_3^2$ | 0.215 | 0.124 | 0.199 | 0.407 |
| CO range | 0.647 | 0.380 | 0.614 | 1.103 |
| NO range | 1.497 | 0.772 | 1.405 | 2.733 |
| NO2 range | 1.334 | 0.706 | 1.236 | 2.473 |

# Coregionalization matrix

**Posterior Median with the associate 95% credible interval (in brackets) of the elements of the coregionalization matrix and the correlation matrix for each location s.**

| $Y_1$ | $Y_2$ | $Y_3$ |
|---|---|---|
| **0.3812** | **0.1108** | **0.1085** |
| **(0.27;0.57)** | **(0.01;0.24)** | **(0.06;0.20)** |
| | **0.7110** | **0.2354** |
| | **(0.47;1.22)** | **(0.14;0.42)** |
| | | **0.3000** |
| | | **(0.21;0.50)** |

# Correlations

| $Y_1$ | $Y_2$ | $Y_3$ |
|---|---|---|
| 1 | 0.2134 | 0.3223 |
| | (0.03;0.41) | (0.17;0.49) |
| | 1 | 0.520 |
| | | (0.31;0.69) |
| | | 1 |

# Prediction of $NO_2$

**Prediction of $NO_2$ based on three different models.**

small

| (i) Independent model for $NO_2$ | | | | | |
|---|---|---|---|---|---|
| Site | Mean | 2.50% | Median | 97.50% | Observed |
| 1 | -4.869 | -5.986 | -4.839 | -3.802 | −4.342 |
| 2 | -4.624 | -5.082 | -4.632 | -4.127 | −4.585 |
| 3 | -4.294 | -4.679 | -4.294 | -3.896 | −4.100 |
| (ii) Model for $NO_2$ conditioned on CO | | | | | |
| 1 | -4.722 | -5.712 | -4.733 | -3.73 | −4.342 |
| 2 | -4.7 | -5.106 | -4.702 | -4.301 | −4.585 |
| 3 | -4.132 | -4.471 | -4.131 | -3.794 | −4.100 |
| (iii) Model for $NO_2$ conditioned on CO and NO | | | | | |
| 1 | -4.5 | -5.313 | -4.508 | -3.679 | −4.342 |
| 2 | -4.585 | -4.964 | -4.587 | -4.22 | −4.585 |
| 3 | -3.966 | -4.26 | -3.964 | -3.653 | −4.100 |

# Other Approaches

- Moving average or kernel convolution of a process:

$$Y_j(\mathbf{s}) = \int k_j(\mathbf{u}) Z(\mathbf{s}+\mathbf{u}) d\mathbf{u} = \int k_j(\mathbf{s}-\mathbf{s}') Z(\mathbf{s}') d\mathbf{s}'$$

  where $Z(\mathbf{s})$ is a univariate spatial process and $k_j$ are kernel functions, $j = 1, 2, \ldots, p$. Yields the cross covariance

$$C_{ij}(\mathbf{s}-\mathbf{s}') = \int \int k_i(\mathbf{s}-\mathbf{s}'+\mathbf{u}) k_j(\mathbf{u}') \rho(\mathbf{u}-\mathbf{u}') d\mathbf{u} d\mathbf{u}'$$

- Convolution of Covariance Functions: Suppose $C_1, C_2, ... C_p$ are valid covariance functions. Define $C_{ij}(\mathbf{s}) = \int C_i(\mathbf{s}-\mathbf{t}) C_j(\mathbf{t}) d\mathbf{t}$. Then the $p \times p$ matrix $C(\mathbf{s}) = \{C_{ij}(\mathbf{s})\}$ is a valid cross covariance function

# Multivariate Areal Data Examples

- Cancer counts for areal units for several different types of cancers
- Employment rates by *sectors* for a set of areal units
- Individual level bivariate data within units, e.g., height adjusted for age (HAZ) and weight adjusted for age (WAZ) with areal unit level spatial effects for each outcome
- Spatially varying coefficient models with coefficients at areal scale because covariates are at areal scale

# Multivariate Areal Data Models

- Now areal units (e.g., counties) instead of points
- Need to model dependence within and across units
- As in univariate case, use spatial random effects $\phi_{ji}$, where again $i = 1, \ldots, n$ indexes region but now $j = 1, \ldots, p$ indexes variables (e.g., cancer type) within region
- Suppose we observe $\mathbf{Y}_i = (Y_{1i}, Y_{21}, ... Y_{pi})$, Then

$$g(E(Y_{ji})) = \mathbf{x}_{ji}^T \boldsymbol{\beta}_j + \phi_{ji} \ ,$$

  with $\boldsymbol{\phi}_i = (\phi_{1i}, \ldots, \phi_{pi})$ and $\boldsymbol{\phi} = (\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n)$.
- Link function $g$ useful for modeling rates (e.g., Poisson disease mapping).
- Multivariate CAR (MCAR) model for the $\phi_{ji}$

# Some models

- Illustrate with $p = 2$
- A disease mapping example: $Y_{1i}$, $Y_{2i}$ are counts for diseases 1 and 2 in unit $i$

$$Y_{ji} \sim Po(\lambda_{ji}), \ , j = 1, 2,$$

$$\lambda_{ji} = E_{ji}\eta_{ji}$$

$$\log\eta_{ji} = \mathbf{X}_{ji}^T \boldsymbol{\beta}_j + \phi_{ji}$$

- Bivariate CAR model for $\{\phi_{1i}, \phi_{2i}\}$
- Height and weight example:

$$\mathbf{Y}_{ir} = \begin{pmatrix} HAZ_{ir} \\ WAZ_{ir} \end{pmatrix} = \mathbf{X}_{ir} \begin{pmatrix} \boldsymbol{\beta}^{(H)} \\ \boldsymbol{\beta}^{(W)} \end{pmatrix} + \begin{pmatrix} \phi_i^{(H)} \\ \phi_i^{(W)} \end{pmatrix} + \begin{pmatrix} \epsilon_{ir}^{(H)} \\ \epsilon_{ir}^{(W)} \end{pmatrix}$$

- Bivariate CAR model for $\{\phi_i^{(H)}, \phi_i^{(W)}\}$

# Multivariate CAR (MCAR) models

- Again, local or neighbor idea, conditioning, CAR
- Approach 1: multivariate CAR (MCAR) in the form $p(\phi_i|\phi_j, j \neq i)$ with

$$p(\phi_i|\phi_{j\neq i}, \Sigma_i) = N\left(\sum_j B_{ij}\phi_j, \Sigma_i\right), \; i = 1, \ldots, n$$

- As earlier, Brook's Lemma yields $p(\phi)$, improper, etc.
- Simplification: $B_{ij} = b_{ij}I, b_{ij} = w_{ij}/w_{i+}, \Sigma_i = (\frac{1}{w_{i+}})\Sigma$
- To make proper, add $\rho$ or perhaps $\rho_j$, $j = 1, \ldots, p$

# cont.

- A coregionalization approach (straightforward)
- With say, $p = 2$, write

$$\begin{pmatrix} \phi_{1i} \\ \phi_{2i} \end{pmatrix} = A \begin{pmatrix} \eta_{1i} \\ \eta_{2i} \end{pmatrix}$$

- $\eta_{1i} \sim \text{CAR}(\tau_1)$, $\eta_{2i} \sim \text{CAR}(\tau_2)$
- $\eta_{1i}$, $\eta_{2i}$ independent